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Maslov's complex germ method and Berry's phase

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Abstract. In the adiabatic approximation the connection of the Berry phase with the quasi-classical trajectory-coherent states of the Schrödinger-type equation (with the arbitrary scalar $\hbar$-(pseudo) differential operator) and the Dirac equation in the external periodic electromagnetic field is studied.

1. Introduction

A traditional problem in the physical literature on the correspondence between the results of classical and quantum mechanics has attracted attention again, due to the discovery of a new quantum effect known as Berry's adiabatic phase [1]. The essence of this phenomenon consists of the following. Let a system undergo an adiabatic evolution by means of a quantum Hamiltonian depending on time by a set of T-periodic functions $R(t)=(R_1(t),\ldots,R_N(t))$. If the system was prepared in some discrete non-degenerate eigenstate $\Phi_n(R(0))$ with the energy $E_n(R(0))$ at the initial moment $t=0$, then the wavefunction of the system $\Psi_n(T)$ will coincide with $\Phi_n(R(0))$ up to a phase factor, so that

$$\Psi_n(T)=\exp(\mathrm{i}\gamma_n(C))\exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t\, E_n(R(t))\right)\Phi_n(R(0)) \tag{1.1}$$

where

$$\gamma_n(C)=-\oint_C \left\langle \Phi_n(R)\middle| \sum_{j=1}^{N}\frac{\partial}{\partial R_j}\Phi_n(R)\right\rangle \mathrm{d}R_j. \tag{1.2}$$

Here C is a closed curve in parametric space $(R_1,\ldots,R_N)$ round which the system is transferred. Hence, in addition to the standard dynamical phase $-\hbar^{-1}\int E\,\mathrm{d}t$, the wave function acquires a new phase term $\gamma_n(C)$—the Berry phase—which is only determined by the set of quantum numbers n associated with the eigenstate $\Phi_n(R)$ and by the counter C.

For the quantum systems corresponding to the bounded integrable Hamiltonian ones, the solution of the problem in the quasi-classical approximation for the Berry phase γ_n is as follows [2]. In this case the system, as a rule, admits a d-parametric family of invariant d tori Λ^d (where d is the dimension of the configuration space).

From this family, the tori giving rise to the quasi-classical spectral series $[\Phi_n, E_n]$ are chosen by means of the Bohr–Sommerfeld–Maslov quantization conditions

$$\frac{1}{2\pi\hbar}\oint_{l_j}\langle p, \mathrm{d}q\rangle=(n_j+\alpha_j) \tag{1.3}$$

where $\{l_j\}$ is a basis of one-cycles on Λ^d, and $\alpha_j=\operatorname{ind} l_j$ are the Maslov indices. The explicit form of the quasi-classical eigenstates Φ_n is given by the Maslov canonical operator with real phase [3]. Then, taking one of such eigenfunctions Φ_n as an initial state of the quantum system, Berry showed that at $\hbar\to 0$ the relation

$$\frac{\partial\gamma_n}{\partial n_j}=-\Delta\theta_j \tag{1.4}$$

holds, where $\Delta\theta_j$ are the Hannay angles whose appearance in classical mechanics is described in [4]. However, one should take into account that the quantization rule (1.3) is, generally speaking, applied only for sufficiently large quantum numbers $n_j\sim 0(\hbar^{-1})$, $\hbar\to 0$, $j=\overline{1, d}$, and, therefore, the asymptotics Φ_n describe highly excited states of a quantum system.

Note that the quasi-classical limit of Berry's phase for integrable quantum systems was discussed in [5–10].

What is more of interest in the present context is to consider the adiabatic evolution of a quantum system in the state corresponding to small quantum numbers and to gain an insight into the corresponding quasi-classical approximation for the Berry phase. Analysis of the well known exactly solvable problems (e.g. see [11]) shows that small quantum numbers are also associated with Lagrangian tori, but with a smaller dimension than that of configuration space. For example, a zero-dimensional torus Λ^0 is a rest point of the Hamiltonian system, and a one-dimensional torus Λ^1 is a closed phase curve which is orbital stable in the linear approximation. A similar situation exists for non-integrable Hamiltonian systems as well, when, as a rule, no family of the d-dimensional Lagrangian tori can be found. Nevertheless, it is often the case that a non-integrable system possessing a certain set of motion integrals permits tori whose dimensions are smaller than d [12].

The general theory of quantization of small-dimensional Lagrangian tori and constructing the corresponding quasi-classical spectral series (the theory of the Maslov complex germ) was developed in [11, 13, 14] (see also [15] for the Dirac equation). The key point of this theory consists of reducing the initial problem of constructing asymptotic solutions to investigation of the classical mechanics equations describing a Lagrangian (to be more precise, isotropic [14]) manifold with a complex germ.

In the present paper we obtain the expression for the Berry phase, generated by the adiabatic motion of a zero-dimensional Lagrangian torus with a complex germ, by means of a special class of localized dynamic states (the so-called quasi-classical trajectory-coherent states [16]). The method proposed makes it possible to consider, from a single point of view, the quantum systems described by the arbitrary $\hbar$ (pseudo) differential scalar operators depending on time t by a set of slowly changing T-periodic functions $R(t)$. The results obtained are extended to the case of the Dirac operator in external T-periodic electromagnetic fields and illustrated by a particular physical example.

2. Quasi-classical spectral series corresponding to zero-dimensional Lagrangian manifolds $\Lambda^0(R)$

Let $\hat{H}(R)=H(-\mathrm{i}\hbar\partial/\partial q, q, R, \hbar)$, $q\in\mathbb{R}^n_q$, be a Weyl-ordered $\hbar$ (pseudo) differential scalar operator depending on $R=(R_1,\ldots,R_N)$ real parameters. The main symbol of the operator $\hat{H}(R)$ is denoted by $\mathscr{H}(p, q, R)=H(p, q, R, 0)$. Consider a spectral problem

$$(\hat{H}(R)-E)\Psi_E(q, R, \hbar)=0. \tag{2.1}$$

It is necessary to construct a special class of asymptotic mod $\mathrm{O}(\hbar^{3/2})$ solutions of equation (2.1) corresponding in the quasi-classical limit $\hbar\to 0$ to a stationary rest point of the classical system described by the Hamiltonian function $\mathscr{H}(p, q, R)$. The main ideas used in constructing such solutions (e.g. see [13]) are as follows.

Let $\Lambda^0(R)=\{p=P_0(R), q=Q_0(R)\}$ be a rest point† of the function $\mathscr{H}(p, q, R)$, i.e. the conditions

$$\mathscr{H}_p(P_0(R), Q_0(R), R)=0 \qquad \mathscr{H}_q(P_0(R), Q_0(R), R)=0 \tag{2.2}$$

are valid. The rest point $\Lambda^0(R)$ is non-degenerate if the matrix

$$\mathscr{H}_{\mathrm{var}}(R)=\begin{pmatrix}-\mathscr{H}_{qp} & -\mathscr{H}_{qq}\\ \mathscr{H}_{pp} & \mathscr{H}_{pq}\end{pmatrix}\Bigg|_{\Lambda^0(R)}$$

is non-singular. Introduce a $2n$-dimensional vector $a(t)=(W(t), Z(t))^{\mathrm{T}}$ and consider the system in variations

$$\dot{a}(t)=\mathscr{H}_{\mathrm{var}}(R)a(t). \tag{2.3}$$

(Here and below a dotted term implies a derivative with respect to t.) The non-degenerate rest point $\Lambda^0(R)$ is called stable in the linear approximation, if all the solutions of equation (2.3) are limited at $t\in(-\infty,\infty)$. Then, if the classical system permits a point of the above type, there exists n linearly independent solutions $a_k(t)$, $k=\overline{1,n}$ of equation (2.3) such that‡

$$\{a_k, \overset{*}{a}_l\}=2\mathrm{i}\delta_{kl}, \qquad \{a_k, a_l\}=0 \qquad k, l=\overline{1,n} \tag{2.4}$$

where

$$a_k(t)=\exp(\mathrm{i}\Omega_k(R)t)a_k(R) \qquad \mathrm{Im}\,\Omega_k(R)=0. \tag{2.5}$$

In other words, a set of number $\Omega_k(R)$ and vectors $a_k(R)=(W_k(R), Z_k(R))^{\mathrm{T}}$ obey the system

$$\mathscr{H}_{\mathrm{var}}(R)a_k(R)=\mathrm{i}\Omega_k(R)a_k(R). \tag{2.6}$$

A complex n-dimensional plane spanned by the vectors $a_k(R)$ is a complex germ at the point $\Lambda^0(R)$ and is denoted by $r^n(\Lambda^0(R))$.

† The rest point Λ^0 is also called a zero-dimensional Lagrangian torus.
‡ Here and below the curly bracket $\{.,.\}$ denotes the antisymmetric inner product.

The vectors $a_k(R)$ and $\overset{*}{a}_k(R)$ of the complex germ are related by the creation and annihilation operators

$$\hat{a}_k^+(R)=\frac{1}{\sqrt{2\hbar}}(\langle \overset{*}{Z}_k(R),\Delta\hat{p}\rangle-\langle \overset{*}{W}_k(R),\Delta q\rangle)$$
$$\hat{a}_k(R)=\frac{1}{\sqrt{2\hbar}}(\langle Z_k(R),\Delta\hat{p}\rangle-\langle W_k(R),\Delta q\rangle) \tag{2.7}$$

where

$$\Delta\hat{p}=-\mathrm{i}\hbar\frac{\partial}{\partial q}-P_0(R) \qquad \Delta q=q-Q_0(R) \tag{2.8}$$

are the operators of small deviations from the equilibrium position $\Lambda^0(R)$. The operators (2.7), owing to relations (2.4), satisfy standard Bose commutation rules

$$[\hat{a}_k,\hat{a}_l^+]=\delta_{kl} \qquad [\hat{a}_k,\hat{a}_l]=[\hat{a}_k^+,\hat{a}_l^+]=0 \qquad k,l=\overline{1,n}. \tag{2.9}$$

Set up square $n\times n$ matrices from the vectors $W_k(R)$ and $Z_k(R)$:

$$B(R)=(W_1(R),\ldots,W_n(R)) \qquad C(R)=(Z_1(R),\ldots,Z_n(R)). \tag{2.10}$$

It follows from equations (2.4) that the matrix $C(R)$ is non-singular. In this way one may determine the symmetric matrix $Q(R)=B(R)C^{-1}(R)$ with a positive-definite imaginary part: $\mathrm{Im}\, Q(R)>0$.

Introduce a vacuum state

$$|0,R\rangle=N_0(\hbar)\frac{1}{\sqrt{\det C(R)}}\exp\left(\frac{\mathrm{i}}{\hbar}S(q,R)\right) \tag{2.11}$$

where the complex phase $S(q,R)$ has the form

$$S(q,R)=\langle P_0(R),\Delta q\rangle+\tfrac{1}{2}\langle\Delta q,Q(R)\Delta q\rangle \tag{2.12}$$

and $N_0(\hbar)=(\pi\hbar)^{-n/4}$ is the normalization factor: $\langle 0,R|0,R\rangle_{L_2}=1$. Now determine a set of functions $|\nu,R\rangle$ as a result of the action of the creation operators $\hat{a}_k^+(R)$ on the vacuum state (2.11):

$$|\nu,R\rangle=\prod_{k=1}^{n}\frac{1}{\sqrt{\nu_k!}}(\hat{a}_k^+)^{\nu_k}|0,R\rangle. \tag{2.13}$$

Then the following statement is true [13].

Proposition 2.1. The functions (2.13) are the asymptotic mod $\mathrm{O}(\hbar^{3/2})$ solutions of the spectral problem (2.1)

$$[\hat{H}(R)-E_\nu(R)]|\nu,R\rangle=\mathrm{O}(\hbar^{3/2}) \tag{2.14}$$

with the eigenvalues

$$E_\nu(R)=E_0(R)+\hbar\sum_{l=1}^{n}\Omega_l(R)(\nu_l+\tfrac{1}{2})+\mathrm{O}(\hbar^2) \tag{2.15}$$

where $E_0(R)=\mathscr{H}(P_0(R), Q_0(R), R)$ and form a complete orthonormal set of states

$$\langle \nu', R | \nu, R\rangle_{L_2}=\delta_{\nu\nu'}. \tag{2.16}$$

The eigenfunctions $|\nu, R\rangle$ and the eigenvalues $E_\nu(R)$ make up a quasi-classical spectral series of the operator $\hat{H}(R)$ corresponding to the zero-dimensional Lagrangian manifold $\Lambda^0(R)$.

3. Trajectory-coherent states and adiabatic Berry's phase

3.1. Statement of the problem

Consider the following evolution equation of the Schrödinger type

$$[-\mathrm{i}\hbar\partial_t+\hat{H}(t)]\Psi(q, t, \hbar)=0 \tag{3.1}$$

where $\hat{H}(t)=H[-\mathrm{i}\hbar\partial/\partial q, q, t, \hbar)$, $q\in\mathbb{R}^n_q$, is an $\hbar$ (pseudo) differential operator (in general, arbitrarily depending on t) with the main symbol $\mathscr{H}(p, q, t)=H(p, q, t, 0)$. On the basis of the Maslov complex germ method [13], for equation (3.1) the asymptotic mod $O(\hbar^{3/2})$ solutions may be constructed in the form of the wave packets—the quasi-classical trajectory-coherent states (TCSs) $\Psi_\nu(q, t, \hbar)=|\nu, t\rangle$ (e.g. see [16]).

Let $r_t=\{p=p(t), q=q(t)\}$ be an arbitrary (but fixed) solution of the canonical system

$$\dot{p}(t)=-\mathscr{H}_q(p, q, t) \qquad \dot{q}(t)=\mathscr{H}_p(p, q, t). \tag{3.2}$$

Quantize system (3.2) in the neighbourhood of the trajectory r_t by the Maslov complex germ method. For this purpose consider the system in variations

$$\dot{a}(t)=\mathscr{H}_{\mathrm{var}}(t)a(t). \tag{3.3}$$

obtained as a result of linearizing the Hamiltonian system (3.2) in a small domain around r_t. Let $a_k(t)=(W_k(t), Z_k(t))^{\mathrm{T}}$, $k=\overline{1, n}$, be a set of n linearly independent complex solutions of equation (3.3) obeying conditions (2.4)†.

Like in the previous section, we obtain complex $n\times n$ matrices $B(t)=(W_1(t), \ldots, W_n(t))$ and $C(t)=(Z_1(t), \ldots, Z_n(t))$ from the components of the vectors $a_k(t)$, where $\det C(t)\neq 0$. For each trajectory r_t there is the vacuum TCS

$$|0, t\rangle=N_0(\hbar)\frac{1}{\sqrt{\det C(t)}}\exp\left\{\frac{\mathrm{i}}{\hbar}S(q, t)\right\} \tag{3.4}$$

with the complex phase

$$S(q, t)=\int_0^t \mathrm{d}t(\langle p(t), \dot{q}(t)\rangle-\mathscr{H}(t))+\langle p(t), \Delta q\rangle+\tfrac{1}{2}\langle \Delta q, Q(t)\Delta q\rangle \tag{3.5}$$

where $N_0(\hbar)=(\pi\hbar)^{-n/4}$, $Q(t)=B(t)C^{-1}(t)$ and $\Delta q=q-q(t)$. By introducing the operator $\Delta\hat{p}=-\mathrm{i}\hbar\partial/\partial q-p(t)$, by analogy with equation (2.7), we may construct the creation $\hat{a}^+_k(t)$ and annihilation $\hat{a}_k(t)$ operators satisfying, as one can easily see, the same commutation rules as in equation (2.9). It is not difficult at all to see the validity of the

† In this case at each fixed t the n-dimensional complex plane spanned by the vectors $a_k(t)$ forms the complex germ at r_t.

conditions $\hat{a}_k(t)|0, t\rangle = 0$, $k=\overline{1,n}$. By means of the creation operators $\hat{a}_k^+(t)$ construct a set of the TCS:

$$|\nu, t\rangle = \prod_{k=1}^{n} \frac{1}{\sqrt{\nu_k!}} (\hat{a}_k^+)^{\nu_k} |0, t\rangle. \tag{3.6}$$

The following results are justified.

Proposition 3.1. Functions (3.6) are the asymptotic mod $O(\hbar^{3/2})$ solutions of equation (3.1), and at each fixed t they form a complete orthonormal set:

$$\langle \nu', t | \nu, t\rangle_{L_2} = \delta_{\nu\nu'}. \tag{3.7}$$

It should be noted that from the condition $\operatorname{Im} S(q, t) \geqslant 0$ follows the localization of states (3.6) in the neighbourhood of the classical trajectory $q = q(t)$—the projection of r_t onto $\mathbb{R}_q^n$.

Following Berry, we now consider the quantum Hamiltonian $\hat{H}(t) = \hat{H}(R(t))$ depending on t through a set of slowly changing T-periodic functions $\{R_1(t), \ldots, R_N(t)\} = R(t)$. Denote the main symbol of the operator $\hat{H}(R(t))$ by $\mathscr{H}(p, q, R(t))$. For the Hamiltonian system (3.2) corresponding to the function $\mathscr{H}(p, q, R(t))$ at the initial moment of time t_0 we take

$$p(t_0) = P_0(R_0) \qquad q(t_0) = Q_0(R_0) \tag{3.8}$$

where $\Lambda^0(R_0) = (P_0(R_0), Q_0(R_0))$, $R_0 = R(t_0)$, is a stationary, in the linear approximation, rest point of the function $\mathscr{H}(p, q, R_0)$. In its turn, for the system in variations (3.3), as initial conditions we choose the set of vectors

$$a_k(t_0) = a_k(R_0) \qquad k = \overline{1, n} \tag{3.9}$$

satisfying conditions (2.4) and (2.6), and forming the complex germ $r^n(\Lambda^0(R_0))$ at the point $\Lambda^0(R_0)$. Thus, the Cauchy data (3.8) and (3.9) define the geometrical object $[\Lambda^0(R_0), r^n(\Lambda^0(R_0))]$—a zero-dimensional Lagrangian manifold with complex germ to which, according to the rules (2.13)–(2.15), the quasi-classical spectral series $[E_\nu(R_0), |\nu, R_0\rangle]$ of the instantaneous Hamiltonian $\hat{H}(R_0)$ corresponds.

Let $[r_t, r^n(r_t)]$ be a solution of the initial value problem (3.8) and (3.9). By its quantization by the complex germ method (see equations (3.4)–(3.6)) we obtain a set of the quasi-classical TCSs $|\nu, t\rangle$. Comparing the explicit form of the functions $|\nu, t\rangle$ and $|\nu, R_0\rangle$, one may make sure of the validity of the equality

$$|\nu, t\rangle|_{t=t_0} = |\nu, R_0\rangle. \tag{3.10}$$

Hence it follows that at each fixed ν the function $|\nu, t\rangle$ (equation (3.6)) is the approximate mod $O(\hbar^{3/2})$ solution of the Cauchy problem (3.10) provided equations (3.8) and (3.9) are valid.

Let us study the Cauchy problem (3.10) in the adiabatic approximation. The solution will be carried out on the assumption that the quasi-classical spectrum (2.14) of the instantaneous Hamiltonian $\hat{H}(R(t))$ is non-degenerate for all the values of t.

3.2. Adiabatic evolution of a classical system

In this subsection the adiabatic solutions of the Hamiltonian system (3.2) with the Hamiltonian function $\mathscr{H}(p, q, R(t))$ and the system in variations (3.3) corresponding to it will be constructed under the adiabatic change of the vector $R(t)$ in the space of the parameters $(R_1, \ldots, R_N)$. The solution will be made in terms of the formal asymptotic power series in a (small) adiabatic parameter T^{-1} (e.g. see [17, 18] where the evolution time T (the period of $R(t)$) is assumed to be rather large†. Define a 'slow time' variable s by $s=t/T$ and put

$$\tilde{R}(s)=R(t)|_{t=sT}. \tag{3.11}$$

We start by considering the Hamiltonian system (3.2) whose solution

$$X(t)=\begin{pmatrix} p(t) \\ q(t) \end{pmatrix}$$

will be examined as

$$X(t)=\overset{0}{X}(s)+\frac{1}{T}\overset{1}{X}(s, \theta)+\mathrm{O}(T^{-2}) \tag{3.12}$$

where $\theta=(T\Phi_1(s), \ldots, T\Phi_n(s))$ is a set of 'rapid' variables in which the real functions $\Phi_k(s)$, $k=\overline{1, n}$, do not depend on T and are to be determined. Calculating the derivative with respect to t by the formula $\partial_t=(1/T)\partial_s+\Phi'(s)\partial_\theta$ and denoting by a prime the derivative with respect to s, we obtain

$$\frac{1}{T}\overset{0}{X}'+\frac{1}{T}\Phi'\overset{1}{X}_{,\theta}=\begin{pmatrix} -\mathscr{H}_q(s) \\ \mathscr{H}_p(s) \end{pmatrix}+\frac{1}{T}\mathscr{H}_{\mathrm{var}}(s)\overset{1}{X}+\mathrm{O}(T^{-2}) \tag{3.13}$$

where

$$\mathscr{H}_q(s)=\mathscr{H}_q(p, q, \tilde{R}(s))_{\Big| \substack{p=\overset{0}{p}(s) \\ q=\overset{0}{q}(s)}}$$

etc. Whence, to zero order we have

$$\mathscr{H}_p(\overset{0}{p}(s), \overset{0}{q}(s), \tilde{R}(s))=0 \qquad \mathscr{H}_q(\overset{0}{p}(s), \overset{0}{q}(s), \tilde{R}(s))=0. \tag{3.14}$$

Thus, the vector

$$\overset{0}{X}(s)=\begin{pmatrix} \overset{0}{p}(s) \\ \overset{0}{q}(s) \end{pmatrix}$$

describes the stationary point $\Lambda^0(\tilde{R}(s))$ of the function $\mathscr{H}(p, q, \tilde{R}(s))$ at each fixed value of s, which, according to the above, implies

$$\begin{aligned} \overset{0}{p}(s)&=P_0(\tilde{R}(s))=P_0(R(t)) \\ \overset{0}{q}(s)&=Q_0(\tilde{R}(s))=Q_0(R(t)). \end{aligned} \tag{3.15}$$

Further, from equations (3.13) and (3.8) in the first approximation the equation for the function $\overset{0}{X}(s, \theta)$ is

$$\Phi'(s)\frac{\partial}{\partial\theta}\overset{1}{X}=\mathscr{H}_{\mathrm{var}}(s)\overset{1}{X}-\overset{0}{X}'(s) \tag{3.16}$$

† It is assumed that the all asymptotics when the Planck constant goes to zero will be true in the adiabatic approximation at $T\to\infty$.

with the initial condition $\overset{1}{X}(s, \theta)|_{s=s_0}=0$, where $s_0=t_0/T$. Let us assume that

$$\overset{1}{X}(s, \theta)=\sum_{k=1}^{n} \{C_k(s, \theta) f_k(s)+\overset{*}{C}_k(s, \theta) \overset{*}{f}_k(s)\} \tag{3.17}$$

where $f_k(s)=a_k(\tilde{R}(s))$ are the eigenvectors of the instantaneous matrix $\mathscr{H}_{\text{var}}(s)$,

$$\mathscr{H}_{\text{var}}(s) f_k(s)=\mathrm{i}\Omega_k(\tilde{R}(s)) f_k(s) \qquad \operatorname{Im} \Omega_k(\tilde{R}(s))=0 \tag{3.18}$$

obeying the relations (2.4). Inserting (3.17) into (3.16) and allowing for the solution obtained to be 2π-periodic with respect to all the rapid variables θ_k† leads to the functions

$$\Phi_k(s)=\int_{s_0}^{s} \Omega_k(\tilde{R}(s))\, \mathrm{d}s \tag{3.19}$$

and the coefficients

$$C_k(s, \theta)=\frac{1}{2\Omega_k(\tilde{R}(s))}\{\overset{*}{f}_k(s), \overset{0}{X}'(s)\}+\tfrac{1}{2} B_k(s) \exp(\mathrm{i}\theta_k). \tag{3.20}$$

Here, $B_k(s)$ are the integration constants which depend on s as a parameter so that the initial condition $C_k(s, \theta)|_{s=s_0}=0$ is fulfilled. (Their explicit form can be found from the following approximation.) As a result we obtain the solution of equation (3.16) in the form

$$\overset{1}{X}(s, \theta)=\begin{pmatrix}\overset{1}{p}(s, \theta)\\ \overset{1}{q}(s, \theta)\end{pmatrix}=\operatorname{Re}\left[\sum_{k=1}^{n} f_k(s)\left(\frac{1}{\Omega_k(\tilde{R}(s))}\{\overset{*}{f}_k(s), \overset{0}{X}'(s)\}+B_k(s) \exp(\mathrm{i}\theta_k)\right)\right]. \tag{3.21}$$

Thus, allowing for equations (3.15) and (3.21) function (3.22) is the approximate mod $O(T^{-2})$ solution of equation (3.2) describing the adiabatic evolution of the classical system.

Now we proceed to construction of the approximate mod $O(T^{-2})$ solutions of the system in variations (3.3). The solution is sought in the form

$$a_k(t)=\overset{0}{a}_k(s, \theta_k)+\frac{1}{T} \overset{1}{a}_k(s, \Xi)+O(T^{-2}) \qquad k=\overline{1, n} \tag{3.22}$$

where $\Xi=(\theta_j, \theta_{lm}, \tilde{\theta}_{lm})$, $j, l, m=\overline{1, n}$, is a set of rapid variables which, apart from the old variables θ_j, includes new ones $\theta_{lm}=T\Phi_{lm}(s)$, $\tilde{\theta}_{lm}=T\tilde{\Phi}_{lm}(s)$. Substituting equations (3.12) and (3.22) into equation (3.3), we get

$$\frac{1}{T} \frac{\partial \overset{0}{a}_k}{\partial s}+\Phi_k'(s) \frac{\partial \overset{0}{a}_k}{\partial \theta_k}+\frac{1}{T} \Phi'(s) \frac{\partial \overset{1}{a}_k}{\partial \Xi}=\mathscr{H}_{\text{var}}(s) \overset{0}{a}_k+\frac{1}{T} \mathscr{H}_{\text{var}}(s) \overset{1}{a}_k+\frac{1}{T} \mathscr{H}_{\text{var}}^{1}(s, \theta) \overset{1}{a}_k+O(T^{-2}) \tag{3.23}$$

where

$$\mathscr{H}_{\text{var}}^{1}(s, \theta)=\langle \overset{1}{p}(s, \theta), \mathscr{H}_{\text{var}, p}(s)\rangle+\langle \overset{1}{q}(s, \theta), \mathscr{H}_{\text{var}, q}(s)\rangle. \tag{3.24}$$

† This additional condition makes it possible to choose a solution (which is of interest for us) from the complete integral.

In the zero-order approximation, in view of equation (3.19), we obtain the equation

$$\Omega_k(\tilde{R}(s))\frac{\partial \overset{0}{a}_k}{\partial \theta_k}=\mathscr{H}_{\text{var}}(s)\overset{0}{a}_k \tag{3.25}$$

and its solution is

$$\overset{0}{a}_k(s,\theta_k)=f_k(s)\exp[\mathrm{i}(\theta_k+\mathscr{N}_k(s))]. \tag{3.26}$$

The solution (3.26) contains the function $\mathscr{N}_k(s)$, which is indeterminate in this approximation, but from condition (3.9) it follows that $\mathscr{N}_k(s_0)=0$. Next, by allowing for equation (3.26) from equation (3.23) we have to the order T^{-1}

$$\Phi'(s)\frac{\partial \overset{1}{a}_k}{\partial \Xi}-\mathscr{H}_{\text{var}}(s)\overset{1}{a}_k=\exp[\mathrm{i}(\theta_k+\mathscr{N}_k(s))]\{\mathscr{H}^{1}_{\text{var}}(s,\theta)f_k-\mathrm{i}\mathscr{N}'_k(s)f_k-f'_k(s)\}. \tag{3.27}$$

By analogy to equation (3.17) the solution of equation (3.27) is chosen in the form

$$\overset{1}{a}_k(s,\Xi)=\sum_{l=1}^{n}\{C^l_k(s,\Xi)f_l(s)+\tilde{C}^l_k(s,\Xi)\overset{*}{f}_l(s)\}. \tag{3.28}$$

We now set

$$\begin{aligned}C^l_k(s,\Xi)&=\exp[\mathrm{i}(\theta_k+\mathscr{N}_k(s))]b^l_k(s,\theta,\theta_{lk})\\ \tilde{C}^l_k(s,\Xi)&=\exp[\mathrm{i}(\theta_k+\mathscr{N}_k(s))]\tilde{b}^l_k(s,\theta,\tilde{\theta}_{lk}).\end{aligned} \tag{3.29}$$

Then from equation (3.27) we obtain the following system of equations which define the coefficients b^l_k and $\tilde{b}^l_k$:

$$\sum_{m=1}^{n}\Omega_m\frac{\partial b^l_k}{\partial\theta_m}+\Phi'_{lk}\frac{\partial b^l_k}{\partial\theta_{lk}}+\mathrm{i}(\Omega_k-\Omega_l)b^l_k=-\frac{\mathrm{i}}{2}(\{\overset{*}{f}_l,f'_k\}+2\mathscr{N}'_k\delta_{kl}-\{\overset{*}{f}_l,\mathscr{H}^{1}_{\text{var}}(s,\theta)f_k\}) \tag{3.30}$$

$$\sum_{m=1}^{n}\Omega_m\frac{\partial \tilde{b}^l_k}{\partial\theta_m}+\tilde{\Phi}'_{lk}\frac{\partial \tilde{b}^l_k}{\partial\tilde{\theta}_{lk}}+\mathrm{i}(\Omega_k+\Omega_l)\tilde{b}^l_k=\frac{\mathrm{i}}{2}(\{f_l,f'_k\}-\{f_l,\mathscr{H}^{1}_{\text{var}}(s,\theta)f_k\}). \tag{3.31}$$

The requirement on the 2π periodicity of the functions b^l_k with respect to all the variables Ξ is, in this case, equivalent to the validity of the following conditions:

$$\Phi_{kl}(s)=\Phi_k(s)-\Phi_l(s) \tag{3.32}$$

and

$$\mathscr{N}_k(s)=-\frac{1}{2}\int_{s_0}^{s}\mathrm{d}s\left\{\overset{*}{f}_k,\left[\frac{\mathrm{d}}{\mathrm{d}s}-\mathscr{H}^{1}_{\text{var}}(\tilde{R}(s))\right]f_k\right\}. \tag{3.33}$$

Here, $\mathscr{H}^{1}_{\text{var}}(\tilde{R}(s))$ denotes a part of the matrix (3.14) which is independent of the variable θ, i.e.

$$\mathscr{H}^{1}_{\text{var}}(s,\theta)=\mathscr{H}^{1}_{\text{var}}(\tilde{R}(s))+\mathrm{Re}\left(\sum_{m=1}^{n}d_m(s)\exp(\mathrm{i}\theta_m)\right). \tag{3.34}$$

The same condition imposed on the coefficients $\tilde{b}^l_k$ leads to the functions $\tilde{\Phi}_{lk}$:

$$\tilde{\Phi}_{lk}(s)=\Phi_k(s)+\Phi_l(s). \tag{3.35}$$

By means of formulae (3.32)–(3.35) it is not difficult to get an explicit form of the functions b_k^l and $\tilde{b}_k^l$:

(i) For $k=l$:

$$b_k^k(s,\theta)=b_k(s)+\sum_{m=1}^{n}\frac{1}{4\Omega_m}[\{\overset{*}{f}_k, d_m(s)f_k\}\exp(\mathrm{i}\theta_k)-\{\overset{*}{f}_k, \overset{*}{d}_m(s)f_k\}\exp(-\mathrm{i}\theta_k)]. \tag{3.36}$$

(ii) For $k\neq l$:

$$\begin{aligned}b_k^l(s,\theta)={}&b_k^l(s)\exp(\mathrm{i}(\theta_l-\theta_k)]\\&+\frac{1}{4}\sum_{m=1}^{n}\left(\frac{\{\overset{*}{f}_l, d_m(s)f_k\}}{\Omega_m+\Omega_k-\Omega_l}\exp(\mathrm{i}\theta_m)-\frac{\{\overset{*}{f}_l, \overset{*}{d}_m(s)f_k\}}{\Omega_m-\Omega_k+\Omega_l}\exp(-\mathrm{i}\theta_m)\right)\\&-\frac{1}{2(\Omega_k-\Omega_l)}\left\{\overset{*}{f}_l, \left[\frac{\mathrm{d}}{\mathrm{d}s}-\mathscr{H}_{\mathrm{var}}^1(\tilde{R}(s))\right]f_k\right\}.\end{aligned} \tag{3.37}$$

(iii) For all k and l:

$$\begin{aligned}\tilde{b}_k^l(s,\theta)={}&\tilde{b}_k^l(s)\exp[-\mathrm{i}(\theta_l+\theta_k)]\\&-\frac{1}{4}\sum_{m=1}^{n}\left(\frac{\{f_l, d_m(s)f_k\}}{\Omega_m+\Omega_k+\Omega_l}\exp(\mathrm{i}\theta_m)-\frac{\{f_l, \overset{*}{d}_m(s)f_k\}}{\Omega_m-\Omega_k-\Omega_l}\exp(-\mathrm{i}\theta_m)\right)\\&+\frac{1}{2(\Omega_k+\Omega_l)}\left\{f_l, \left[\frac{\mathrm{d}}{\mathrm{d}s}-\mathscr{H}_{\mathrm{var}}^1(\tilde{R}(s))\right]f_k\right\}.\end{aligned} \tag{3.38}$$

To define the functions $b_k^l(s)$, $\tilde{b}_k^l(s)$ appearing in equations (3.36)–(3.38) we need the following order of T^{-1}. Due to equation (3.9) their initial values $b_k^l(s_0)$ and $\tilde{b}_k^l(s_0)$ are found from the conditions $b_k^l(s,\theta)|_{s=s_0}=\tilde{b}_k^l(s,\theta)|_{s=s_0}=0$.

The following should be noted in conclusion. The assumption on non-degeneration of the quasi-classical spectrum (2.14) made at the end of section 3.1 implies that no resonance relations $\Sigma_{m=1}^{n} l_k\Omega_k=0(l_1,\ldots,l_n$ are integers) exist between the frequencies $\Omega_1,\ldots,\Omega_n$. This condition was mainly used when deriving formulae (3.36)–(3.38).

3.3. Berry's phase

Now we return to the adiabatic solution of the Cauchy problem (3.10). It follows from the previous section that:

(i) For the solution of the Hamiltonian system (3.2) which obeys the initial condition (3.8), according to equations (3.12) and (3.15), we obtain

$$\begin{aligned}p(t)&=P_0(R(t))+\mathrm{O}(T^{-1})\\q(t)&=Q_0(R(t))+\mathrm{O}(T^{-1}).\end{aligned} \tag{3.39}$$

(ii) The solutions of the system in variations (3.3) obeying the initial conditions (3.9), as follows from equations (3.22), (3.26) and (3.33), have the form

$$a_k(t)=a_k(R(t))\exp\left(\mathrm{i}\int_{t_0}^{t}\mathrm{d}t\,\Omega_k(R(t))-\mathrm{i}\int_{t_0}^{t}\mathrm{d}t\,\Omega_k^1(R(t))\right)+\mathrm{O}(T^{-1}) \tag{3.40}$$

where

$$\Omega_k^1(R(t))=\frac{1}{2}\left\{\overset{*}{a}_k(R(t)),\left(\frac{\mathrm{d}}{\mathrm{d}t}-\mathscr{H}_{\mathrm{var}}^1(R(t))\right)a_k(R(t))\right\}. \tag{3.41}$$

The matrix $\mathscr{H}_{\mathrm{var}}^1(R(t))$ appearing here was defined in equation (3.34) and, according to formulae (3.21) and (3.24) may be represented as

$$\mathscr{H}_{\mathrm{var}}^1(R(t))=\langle\nabla\mathscr{H}_{\mathrm{var}}(R(t)),\overset{1}{X}(R(t))\rangle \tag{3.42}$$

where $\nabla\equiv(\partial/\partial p,\partial/\partial q)^{\mathrm{T}}$, and the vector $\overset{1}{X}(R(t))$ is

$$\overset{1}{X}(R(t))=\sum_{k=1}^{n}\frac{1}{\Omega_k(R(t))}\,\mathrm{Re}\left(a_k(R(t))\left\{\overset{*}{a}_k(R(t)),\frac{\mathrm{d}}{\mathrm{d}t}\overset{0}{X}(R(t))\right\}\right). \tag{3.43}$$

By substituting expressions (3.39) and (3.40) into equation (3.6) we obtain after simple calculations

$$|\nu,t\rangle=|\nu,R(t)\rangle\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\mathrm{d}t\,E_\nu(R(t))+\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\langle P_0(R(t)),\mathrm{d}Q_0(R(t))\rangle\right.$$
$$\left.+\mathrm{i}\sum_{k=1}^{n}\int_{t_0}^{t}\mathrm{d}t\,\Omega_k^1(R(t))(\nu_k+\tfrac{1}{2})\right)+\mathrm{O}(T^{-1}). \tag{3.44}$$

Whence it follows that, if a quantum system at an initial moment of time $t=t_0$ is in its eigenstate $|\nu,R_0\rangle$ corresponding to the energy level $E_\nu(R_0)$, then, during the time T with the adiabatic change of the vector $R(t)$, where $R(t+T)=R(t)$, a system returns to its initial state by acquiring an additional phase. By comparing (3.44) and (1.1) we obtain for the Berry phase

$$\gamma_\nu(T)=\frac{1}{\hbar}\int_0^T\langle P_0(R(t)),\mathrm{d}Q_0(R(t))\rangle+\sum_{k=1}^{n}\int_0^T\mathrm{d}t\,\Omega_k^1(R(t))(\nu_k+\tfrac{1}{2}). \tag{3.45}$$

To emphasize a purely geometrical meaning of phase (3.45) we rewrite it as follows:

$$\gamma_\nu(T)=\gamma_\nu(C)=\frac{1}{\hbar}\oint_C\left\langle P_0(R),\frac{\partial Q_0(R)}{\partial R_l}\right\rangle\mathrm{d}R_l+\sum_{k=1}^{n}\oint_C\{\overset{*}{a}_k(R),\mathscr{T}_k^{(l)}(R)\}(\nu_k+\tfrac{1}{2})\,\mathrm{d}R_l \tag{3.46}$$

where, due to formulae (3.41)–(3.43)

$$\mathscr{T}_k^{(l)}(R)=\frac{1}{2}\frac{\partial a_k(R)}{\partial R_l}-\frac{1}{2}\sum_{l=1}^{n}\frac{1}{\Omega_l(R)}\left[\mathrm{Re}\left(\langle\nabla\mathscr{H}_{\mathrm{var}}(R),a_l(R)\rangle\left\{\overset{*}{a}_l(R),\frac{\partial\overset{0}{X}(R)}{\partial R_l}\right\}\right)\right]a_k(R). \tag{3.47}$$

Here C is a circuit round which the end of the vector $R(t)$, $t\in[0,T]$, moves in the space of parameters $(R_1,\ldots,R_N)$.

As may be inferred from equations (3.46) and (3.47) the Berry adiabatic phase $\gamma_\nu(C)$ is completely defined by the two following (caused by the classical motion) geometrical functions:

(i) The trajectory $\Lambda^0(R)=(P_0(R),Q_0(R))$, $R\in C$, consisting of stable (in the linear approximation) rest points of the Hamiltonian system.

(ii) The complex germ $r^A(\Lambda^0(R)), R \in C$, consisting of n linearly independent eigenvectors $a_k(R)$, $k=\overline{1,n}$, of the matrix $\mathscr{H}_{\rm var}(R)$ and normalized by condition (2.4).

4. Berry's phase for a generalized harmonic oscillator

We shall illustrate the results obtained above by the classical example of a one-dimensional generalized harmonic oscillator described by the Hamiltonian

$$\hat{H}(t)=\tfrac{1}{2}[\mu(t)\hat{p}^2+\sigma(t)\hat{q}^2+\rho(t)(\hat{p}\hat{q}+\hat{q}\hat{p})] \tag{4.1}$$

where $R=(\mu, \sigma, \rho)$ are the parameters specifying the adiabatic evolution of the quantum system. The corresponding classical Hamiltonian is

$$\mathscr{H}(p, q, R(t))=\tfrac{1}{2}(\mu(t)p^2+\sigma(t)q^2+2\rho(t)pq). \tag{4.2}$$

At $\rho^2 \neq \mu\sigma$, function (4.2) has only the rest point $\Lambda^0=(P_0=0, Q_0=0)$. The requirement for the point Λ^0 to be stationary in the linear approximation results in an additional condition $\rho^2<\mu\sigma$. In this case the spectral problem for the matrix $\mathscr{H}_{\rm var}(R)$ permits the following solution:

$$a(R)=\frac{1}{\sqrt{\mu\Omega(R)}}\begin{pmatrix}-\rho+\mathrm{i}\Omega(R)\\ \mu\end{pmatrix} \tag{4.3}$$

where $\Omega(R)=\zeta\sqrt{\mu\sigma-\rho^2}$, $\zeta=\operatorname{sgn}\mu$, and $\{a(R), \overset{*}{a}(R)\}=2\mathrm{i}$. But then for the Berry phase according to equations (3.46) and (3.47) we obtain the value

$$\gamma_\nu(C)=\tfrac{1}{2}(\nu+\tfrac{1}{2})\oint_C\left\{\overset{*}{a}(R), \frac{\partial a(R)}{\partial R_i}\right\}\mathrm{d}R_i=(\nu+\tfrac{1}{2})\oint_C\frac{1}{2\sqrt{\mu\sigma-\rho^2}}\left(\mathrm{d}\rho-\frac{\rho}{\mu}\,\mathrm{d}\mu\right) \tag{4.4}$$

coinciding with the result of [2]. A similar method of deriving formula (4.4), but based on the correlated coherent states was proposed in [19].

5. Berry's adiabatic phase for the Dirac wavefunction

Now consider the Hamiltonian

$$\hat{H}_{\rm D}(R)=c\langle\boldsymbol{\alpha}, \boldsymbol{P}\rangle+\rho_3 mc^2+eA_0 \tag{5.1}$$

describing a Dirac particle in an electromagnetic field with potential $A_0(\boldsymbol{x}, R)$, $\boldsymbol{A}(\boldsymbol{x}, R)$ depending on N parameters $(R_1, \ldots, R_N)=R$. Here $\hat{\boldsymbol{P}}=\hat{\boldsymbol{p}}-(e/c)\boldsymbol{A}(\boldsymbol{x}, R)$, $\hat{\boldsymbol{p}}=-\mathrm{i}\hbar\partial_{\boldsymbol{x}}$ are the kinetic and generalized momenta operators, $e=-e_0$ is the electron charge, and $\boldsymbol{\alpha}=\rho_1\boldsymbol{\Sigma}$, ρ_3 are the Dirac matrices in the standard representation. The symbol of the operator (5.1) is the Hermitian matrix of the form $\mathscr{H}_{\rm D}(\boldsymbol{p}, \boldsymbol{x}, R)=c\langle\boldsymbol{\alpha}, \boldsymbol{P}\rangle+\rho_3 mc^2+eA_0$, where $\boldsymbol{P}=\boldsymbol{p}-(e/c)\boldsymbol{A}$. The spectral problem for the matrix $\mathscr{H}_{\rm D}$,

$$\mathscr{H}_{\rm D}\Pi_\pm=\lambda^{(\pm)}\Pi_\pm \tag{5.2}$$

has the following solution [20]:

$$\lambda^{(\pm)}(p, x, R) = eA_0(x, R) \pm \varepsilon \qquad \varepsilon = (c^2 P^2 + m^2 c^4)^{1/2} \tag{5.3}$$

$$\Pi_+(p, x, R)) = \frac{1}{\sqrt{2(1+\gamma^{-1})}} \begin{pmatrix} 1+\gamma^{-1} \\ \sigma\beta \end{pmatrix}$$
$$\Pi_-(p, x, R)) = \frac{1}{\sqrt{2(1+\gamma^{-1})}} \begin{pmatrix} \sigma\beta \\ -(1+\gamma^{-1}) \end{pmatrix} \tag{5.4}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, $\beta = (c/\varepsilon)P$, $\gamma^{-1} = (1-\beta^2)^{1/2}$. Matrices (5.4) satisfy the orthonormality and completeness relations: $\overset{+}{\Pi}_\zeta, \Pi_{\zeta'} = \delta_{\zeta,\zeta'}, \Sigma_\zeta \Pi_\zeta \overset{+}{\Pi}_\zeta = 1$, $\zeta = \pm 1$. Let the function $\lambda^{(+)}(p, x, R)$ possess at each fixed R a stationary, in the linear approximation, rest point $\Lambda^0(p_0(R), x_0(R))$, then the complex germ $r^3(\Lambda^0(R))$ corresponding to it is formed by the vectors (2.4)–(2.6).

Denote the magnitude of the magnetic field at the point $x = x_0(R)$ by $H(R)$ and consider the following two-component spinor problem:

$$\langle \sigma, \mathscr{B}(R) \rangle v_\zeta(R) = \Omega_\zeta(R) v_\zeta(R) \tag{5.5}$$

where $\mathscr{B}(R) = (e_0/2mc)H(R)$ is the 'polarization' vector at the rest point. Assuming $H(R)/|H(R)| = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$ it is not difficult to obtain a general solution of equation (5.5). Really, $\Omega_\zeta(R) = \zeta|\mathscr{B}(\tau)|$ are the eigenvalues with the eigenvectors

$$v_\zeta(R) = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta\sqrt{1+\zeta\cos\theta}\, e^{-1\varphi/2} \\ \sqrt{1-\zeta\cos\theta}\, e^{i\varphi/2} \end{pmatrix} \qquad \zeta = \pm 1 \tag{5.6}$$

obeying the orthonormality and completeness relations: $\overset{+}{v}_\zeta, v_{\zeta'} = \delta_{\zeta,\zeta'}, \Sigma_\zeta v_\zeta \overset{+}{v}_\zeta = 1$.

Now we introduce the following:

(i) The Weyl-ordered $\hbar$ pseudodifferential operator $\hat{\lambda}^+(R) = \lambda^{(+)}(\hat{p}, x, R)$ with the Hamiltonian function $\lambda^{(+)}(p, x, \mathrm{R})$, and construct the quasi-classical spectral series $[E_\nu(R), |\nu, R\rangle]$ corresponding to the rest point $\Lambda^0(R)$ for it (see section 2).

(ii) The operator $\hat{Q}_k(R)$, $k = 1, 2$, defined as

$$\hbar^{k/2} \hat{Q}_k(R) = -\frac{c}{k!} \langle \sigma, \hat{\delta}^k P(R) \rangle \tag{5.7}$$

where $\hat{\delta}^k P(R)$ implies the kth term in the Taylor power expansion over the operators $\Delta\hat{p} = \hat{p} - p_0(R)$, $\Delta x = x - x_0(R)$ in the neighbourhood of $\Lambda^0(R)$.

(iii) The matrices $\Pi_\zeta(R) = \Pi_\zeta(p, x, R)|_{\Lambda^0(R)}$ which, due to the condition $\beta(R) = (1/c)\dot{x}(R) = 0$, are likely to be equal to

$$\Pi_+(R) = \frac{1}{\sqrt{2}} \begin{pmatrix} E_{2\times 2} \\ O_{2\times 2} \end{pmatrix} \qquad \Pi_-(R) = \frac{1}{\sqrt{2}} \begin{pmatrix} O_{2\times 2} \\ -E_{2\times 2} \end{pmatrix}. \tag{5.8}$$

Then, the following statement is true.

Proposition 5.1. If

$$E_{\nu,\zeta}(R) = E_\nu(R) + \hbar\Omega_\zeta(R) + O(\hbar^2) \tag{5.9}$$

then the functions

$$\Psi_{E_{\nu,\zeta}(R)}(x, \hbar) = \left(\Pi_+(R) + \frac{1}{2mc^2} \Pi_-(R) \sum_{k=1}^{2} \hbar^{k/2} \hat{Q}_k(R) \right) v_\zeta(R) |\nu, R\rangle \tag{5.10}$$

localized in the neighbourhood of $\Lambda^0(R)$ are the asymptotic mod $O(\hbar^{3/2})$ eigenfunctions of the operator $\hat{H}_D(R)$,

$$[\hat{H}_D(R)-E_{\nu,\zeta}(R)]\Psi_{E_{\nu,\zeta}(R)}(x,\hbar)=O(\hbar^{3/2}) \tag{5.11}$$

and form with an accuracy $O(\hbar^{1/2})$ a complete orthonormal set of states

$$\langle\Psi_{E_{N'}},\Psi_{E_N}\rangle_D=\int d^3x\,\overset{+}{\Psi}_{E_{N'}}\Psi_{E_N}=\delta_{N'N}+O(\hbar^{1/2}). \tag{5.12}$$

The sequence $[E_{\nu,\zeta}(R),\Psi_{E_{\nu,\zeta}}(R)]$ introduced in this way is the quasi-classical spectral series of the Dirac operator $\hat{H}_D(R)$ corresponding to the zero-dimensional Lagrangian torus $\Lambda^0(R)$.

Now we turn to the discussion of the case when the Dirac operator $\hat{H}_D(R(t))$ (5.1) depends on N slowly changing T-periodic functions of time $(R_1(t),\ldots,R_N(t))=R(t)$. It is necessary to construct the approximate mod $O(\hbar^{3/2})$ solution of the equation

$$[-i\hbar\partial_t+\hat{H}_D(R(t))]\Psi_D(x,t,\hbar)=O(\hbar^{3/2}) \tag{5.13}$$

obeying the initial condition

$$\Psi_D(x,t,\hbar)|_{t=t_0}=\Psi_{E_{\nu,\zeta}(R_0)}(x,\hbar) \tag{5.14}$$

where on the right-hand side of equation (5.14) one of the asymptotic eigenfunctions (5.10) of the instantaneous Hamiltonian $\hat{H}_D(R_0)=\hat{H}_D(R(t_0))$ is chosen

The solution of this problem is expressed by means of the following:

(i) The phase trajectory $r_t=\{p=p(t),\ x=x(t)\}$—the solution of the Hamiltonian system (3.2) with the Hamiltonian $\lambda^{(+)}(p,x,R(t))$ which satisfies the initial condition $r_t(t=t_0)=\Lambda^0(R_0)=(p_0(R_0),x_0(R_0))$.

(ii) The function $|\nu,t\rangle$—the approximate mod $O(\hbar^{3/2})$ solution of the Cauchy problem (3.10) for the equation

$$[-i\hbar\partial_t+\hat{\lambda}^+(R(t))]|\nu,t\rangle=O(\hbar^{3/2}). \tag{5.15}$$

(iii) The spinor $v_\zeta(t)$—the solution of the Cauchy problem for the spin-only equation

$$\left(-i\frac{d}{dt}+\langle\sigma,\mathscr{B}(t)\rangle\right)v_\zeta(t)=0 \tag{5.16}$$

$$v_\zeta(t)|_{t=t_0}=v_\zeta(R_0) \tag{5.17}$$

with the initial spinor $v_\zeta(R_0)$ (equation (5.6)) and the 'polarization' vector

$$\mathscr{B}(R)=\frac{e_0c}{2\varepsilon(t)}\left(H(x,R(t))-\frac{\dot{x}(t)\times E(x,R(t))}{c(1+\gamma^{-1}(t))}\right)\Bigg|_{x=x(t)} \tag{5.18}$$

where H and E are the magnetic and electric components of an external electromagnetic field.

(iv) The operators $\hat{Q}_k(t)$, $k=1,2$, by setting

$$\hat{Q}_1(t)=\frac{2}{\sqrt{\hbar}}\left(\langle\sigma,\dot{x}\rangle\frac{\langle\dot{x},\hat{\delta}^1P(t)\rangle}{c^2(1+\gamma^{-1})}-\langle\sigma,\hat{\delta}^1P(t)\rangle\right) \tag{5.19}$$

$$\hat{Q}_2(t)=\frac{2}{\sqrt{\hbar}}\left(\langle\sigma,\dot{x}\rangle\frac{\langle\dot{x},\hat{\delta}^2P(t)\rangle}{c^2(1+\gamma^{-1})}-\langle\sigma,\hat{\delta}^2P(t)\rangle\right)+\frac{i}{2c}\left(\langle\sigma,\dot{x}\rangle\frac{\gamma\langle\dot{x},\ddot{x}\rangle}{c^2(1+\gamma^{-1})}+\langle\sigma,\ddot{x}\rangle\right)$$
$$-\frac{1}{\sqrt{\hbar}\varepsilon(t)}\langle\dot{x},\hat{\delta}^1P(t)\rangle\hat{Q}_1(t). \tag{5.20}$$

Here $\hat{\delta}^k \boldsymbol{P}(t)$, $k=1, 2$, denotes the kth term in the Taylor power expansion over the operators $\Delta\hat{\boldsymbol{p}}=\hat{\boldsymbol{p}}-\boldsymbol{p}(t)$ and $\Delta \boldsymbol{x}=\boldsymbol{x}-\boldsymbol{x}(t)$ in the neighbourhood of the trajectory r_t.

Then, as follows from the results of [20], the function

$$\Psi_{\mathrm{D}_{\nu,\zeta}}(\boldsymbol{x}, t, \hbar)=\left[\Pi_+(t)+\frac{1}{2\varepsilon(t)}\Pi_-(t)\sum_{k=1}^{2}\hbar^{k/2}\hat{Q}_k(t)\right]v_\zeta(t)|\nu, t\rangle \tag{5.21}$$

is the asymptotic mod $O(\hbar^{3/2})$ solution of the Cauchy problem (5.13) and (5.14).

Now we obtain the asymptotic expansion of the state (5.21) in terms of the small parameter T^{-1} (with an accuracy $O(T^{-1})$), allowing for the adiabatic change of the vector $R(t)$. As in the scalar case, we assume that the Dirac operator $\hat{H}_{\mathrm{D}}(R(t))$ has the non-degenerate quasi-classical energy spectrum at each fixed t.

First of all, making use of the asymptotic formulae (3.39) it is easy to verify that the expression enclosed in the brackets in equation (5.21) with an accuracy to $O(T^{-1})$ coincides with the corresponding one of equation (5.10). Since the formula describing the adiabatic evolution of the state $|\nu, t\rangle$ was obtained by us earlier (see equation (3.44)), the task of constructing the adiabatic approximation for the Dirac function (5.21) is reduced to deriving the asymptotic in T^{-1} solutions of equation (5.16). We provide this in the appendix. Ultimately, we arrive at the following result:

$$v_\zeta(t)=v_\zeta(R(t))\exp\left(-\mathrm{i}\int_{t_0}^{t}\mathrm{d}t\,\Omega_\zeta(R(t))+\mathrm{i}\int_{t_0}^{t}\mathrm{d}t\,\Omega_\zeta^1(R(t))\right)+O(T^{-1}) \tag{5.22}$$

where

$$\Omega_\zeta^1(R(t))=\overset{+}{v}_\zeta(R(t))\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}-\langle\boldsymbol{\sigma}, \mathscr{B}^1(R(t))\rangle\right)v_\zeta(R(t)). \tag{5.23}$$

Here, the vector $\mathscr{B}^1(R(t))$ is equal to

$$\mathscr{B}^1(R(t))=\frac{e_0}{2mc}\left[-\frac{1}{c}\frac{\mathrm{d}\boldsymbol{x}_0(R(t))}{\mathrm{d}t}\times \boldsymbol{E}(R(t))\right.$$
$$\left.+\sum_{k=1}^{3}\frac{1}{\Omega_k(\mathrm{R(t)})}\mathrm{Re}\left(\langle\nabla \boldsymbol{H}(R(t)), a_k(R(t))\rangle\left\{\overset{*}{a}_k(R(t)), \frac{\mathrm{d}\overset{0}{X}(R(t))}{\mathrm{d}t}\right\}\right)\right]. \tag{5.24}$$

Whence, in view of equations (3.44) and (5.22), we have

$$\Psi_{\mathrm{D}_{\nu,\zeta}}(\boldsymbol{x}, t, \hbar)=\Psi_{E_{\nu,\zeta(R(t))}}(\boldsymbol{x}, \hbar)\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\mathrm{d}t\,E_{\nu,\zeta}(R(t))+\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\langle \boldsymbol{p}_0(R(t)), \dot{\boldsymbol{x}}_0(R(t))\rangle\,\mathrm{d}t\right.$$
$$\left.+\mathrm{i}\sum_{k=1}^{3}\int_{t_0}^{t}\mathrm{d}t\,\Omega_k^1(R(t))(\nu+\tfrac{1}{2})+\int_{t_0}^{t}\mathrm{d}t\,\Omega_\zeta^1(R(t))\right)+O(T^{-1}) \tag{5.25}$$

where $E_{\nu,\zeta}(R(t))$ is the quasi-classical energy level (5.9) of the instantaneous Hamiltonian $\hat{H}_{\mathrm{D}}(R(t))$ corresponding to the eigenfunctions $\Psi_{E_{\nu,\zeta(R(t))}}(\boldsymbol{x}, \hbar)$ (equation (5.10)).

If we denote now

$$\gamma_\zeta(C)=\oint_C \overset{+}{v}_\zeta(R)\mathscr{T}_\zeta^{(i)}(R)\,\mathrm{d}R_i \tag{5.26}$$

where

$$\mathcal{F}_\zeta^{(i)}(R) = \mathrm{i}\,\frac{\partial v_\zeta(R)}{\partial R_i} + \frac{e_0}{2mc}\Bigg[\frac{1}{c}\,\frac{\partial x_0(R)}{\partial R_i} \times E(R) - \sum_{k=1}^{3}\frac{1}{\Omega_k(\mathrm{R})}\,\mathrm{Re}\Bigg(\langle \nabla\,\hat{H}(R), a_k(R)\rangle \Bigg\{\overset{*}{a}_k(R), \frac{\partial \overset{0}{X}(R)}{\partial R_i}\Bigg\}\Bigg)\Bigg] v_\zeta(R) \tag{5.27}$$

then by comparing equations (1.1) and (5.25), in conjunction with (3.46) and (5.26), we immediately obtain the expression for the Berry phase:

$$\gamma_{\mathrm{D}}(C) = \gamma_\nu(C) + \gamma_\zeta(C). \tag{5.28}$$

So, in the quasi-classical trajectory-coherent approximation we have the following. During the process of the adiabatic evolution along a closed curve C in parameter space the Dirac wavefunction Ψ_{D} acquires the Berry phase, which consists of two parts. One of them, $\gamma_\nu(C)$, *is induced by the adiabatic motion of the geometrical object* $[\Lambda^0(R), r^3(\Lambda^0(R))]$, $R\in C$, and is determined by the scalar part of Ψ_{D}, i.e. the function $|\nu, t\rangle$. The second term, $\gamma_\zeta(C)$, is due to the adiabatic transport of the spinor $v_\nu(R)$ around C and is associated with the spinor part of Ψ_{D}.

6. Berry's phase of the Dirac particle in the external periodic electromagnetic field

Let the electromagnetic field be given by the potentials

$$A_0(t) = \tfrac{1}{2}\mu(t) r^2 \qquad A(t) = \tfrac{1}{2}H(t)\times r \tag{6.1}$$

where r is the radius vector, and $\mu(t) < 0$, $H(t)$ is a set of arbitrary T-periodic functions specifying the adiabatic evolution of the quantum system. Thus, in this case an instantaneous state of the quantum system is characterized by a set of the parameters $R = (\mu, H)$.

For the potentials (6.1) the magnetic component of the electromagnetic field coincides with the vector $H(t)$, and the electrical field is equal to $E(t) = -(1/2c)\dot{H}(t)\times r - \mu(t) r$. The classical motion of an electron is described by the system of equations

$$\dot{r} = \frac{c^2}{\varepsilon(t)} P \qquad \dot{p} = -e\mu(t) r + \frac{ec}{2\varepsilon(t)} P\times H(t) \tag{6.2}$$

where $P = p - (e/c)A(t)$, $\varepsilon(t) = (c^2P^2 + m^2c^4)^{1/2}$. From here it follows that only a single rest point $\Lambda^0 = (p_0 = 0, r_0 = 0)$ is possible.

The set of equations (2.6) defining the eigenvectors $a(R) = (W(R), Z(R))^{\mathrm{T}}$ of the matrix $\mathcal{H}_{\mathrm{var}}(R)$ at the point Λ^0 is conveniently written in the form

$$\frac{e}{2mc} W\times H - \left(e\mu + \frac{e^2}{4mc^2}H^2\right)Z + \frac{e^2}{4mc^2}\langle H, Z\rangle H = \mathrm{i}\Omega W \tag{6.3}$$

$$W = \mathrm{i}m\Omega Z + \frac{e}{2c} H\times Z. \tag{6.4}$$

Inserting equation (6.4) into equation (6.3) we get the equation for $\boldsymbol{Z}$:

$$\frac{\mathrm{i}e\Omega}{c}\boldsymbol{H}\times\boldsymbol{Z}=\left(\Omega^2-\frac{e\mu}{m}\right)\boldsymbol{Z}. \tag{6.5}$$

In terms of the spherical polar representation $\boldsymbol{H}=(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$ the solutions of equations (6.3)–(6.5) have the form

$$a_0(R)=\begin{pmatrix}\mathrm{i}\sqrt{m\Omega_1(R)}\,\boldsymbol{n}_r(R)\\ \dfrac{1}{\sqrt{m\Omega_1(R)}}\boldsymbol{n}_r(R)\end{pmatrix}\qquad \Omega_1(R)=(e\mu/m)^{1/2} \tag{6.6}$$

$$a_\eta(R)=\begin{pmatrix}\dfrac{\mathrm{i}}{2N(R)}(\boldsymbol{n}_\theta(R)-\mathrm{i}\eta\boldsymbol{n}_\varphi(R))\\ N(R)(\boldsymbol{n}_\theta(R)-\mathrm{i}\eta\boldsymbol{n}_\varphi(R))\end{pmatrix}\qquad \eta=\pm1$$

$$\Omega_\eta=-\eta er/2mc+\sqrt{e\mu/m+e^2r^2/4m^2c^2} \tag{6.7}$$

where $N(R)=(e\mu/m+e^2r^2/4m^2c^2)^{-1/4}$, and $(\boldsymbol{n}_r, \boldsymbol{n}_\theta, \boldsymbol{n}_\varphi)$ is the spherical orthonormal frame. Hence, for the Berry phase $\gamma_\nu(C)$ (equation (3.46) (induced by the adiabatic motion of the complex germ) the expression

$$\gamma_\nu(C)=-\sum_\eta \eta(\nu_\eta+\tfrac{1}{2})\oint_C \langle\boldsymbol{n}_\theta(R), \mathrm{d}\boldsymbol{n}_\varphi(R)\rangle \tag{6.8}$$

follows. It can also be written as a surface integral by the use of the Stokes theorem $(\partial\Sigma=C)$

$$\begin{aligned}\gamma_\nu(C)&=-\sum_\eta \eta(\nu_\eta+\tfrac{1}{2})\iint_\Sigma \langle\mathrm{d}\boldsymbol{n}_\theta(R)\wedge\mathrm{d}\boldsymbol{n}_\varphi(R)\rangle\\ &=(\nu_-+\nu_+)\iint_\Sigma \frac{1}{2|\boldsymbol{H}|^3}\sum_{i,j,k}^{3}\varepsilon^{ijk}H_i\mathrm{d}H_j\wedge\mathrm{d}H_k.\end{aligned} \tag{6.9}$$

Now we look for a 'spin' part of the Berry phase $\gamma_\zeta(C)$. The eigenvectors of the matrix $\boldsymbol{\sigma}\mathscr{B}(R)$, where in our case $\mathscr{B}(R)=(e_0/mc)\boldsymbol{H}$, due to formula (5.6), have the form

$$v_\zeta(R)=\frac{1}{\sqrt{2}}\begin{pmatrix}\zeta\sqrt{1+\zeta\cos\theta}\,\mathrm{e}^{-\mathrm{i}\varphi(R)/2}\\ \sqrt{1+\zeta\cos\theta}\,\mathrm{e}^{\mathrm{i}\varphi(R)/2}\end{pmatrix}\qquad \zeta=\pm1. \tag{6.10}$$

Whence, by means of formula (5.26) we find

$$\gamma_\zeta(C)=-\frac{\zeta}{2}\iint_{\Sigma(\partial\Sigma=C)}\frac{1}{2|\boldsymbol{H}|^3}\sum_{i,j,k}^{3}\varepsilon^{ijk}H_i\mathrm{d}H_j\,\mathrm{d}H_k. \tag{6.11}$$

It follows from equations (6.9) and (6.11) that the overall Berry phase of the Dirac electron is equal to

$$\gamma_\mathrm{D}(C)=\gamma_\nu(C)+\gamma_\zeta(C)=\left(\nu_-+\nu_+-\frac{\zeta}{2}\right)\iint_{\Sigma(\partial\Sigma=C)}\mathrm{d}\Sigma\,\frac{\boldsymbol{H}}{|\boldsymbol{H}|^3}. \tag{6.12}$$

The integral on the right-hand side of equation (6.12) is the Gauss integral, which is equal to the solid angle $\Omega(C)$ of the circuit C as viewed from the origin $\boldsymbol{H}=0$. Therefore, we finally have

$$\gamma_{\mathrm{D}}(C)=(\nu_{-}+\nu_{+}-\zeta/2)\Omega(C). \tag{6.13}$$

In conclusion, we note that the value $\gamma_{\mathrm{D}}(C)$ differs from zero only on the condition that all the components of the magnetic field $\boldsymbol{H}(t)$ are not equal to zero. In obtaining equation (6.13) $\mu(t)\neq 0$, since otherwise (see equation (6.6)) the initial assumption of the non-degeneration of the quasi-classical spectrum (5.9) is violated.

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Appendix

We are going to find the approximation mod $\mathrm{O}(T^{-2})$ solution of the Cauchy problem (5.17). In other words, we need to construct the function $v_\zeta(t)$ obeying the equation

$$\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}+\langle\boldsymbol{\sigma},\mathscr{B}(t)\rangle\right)v_\zeta(t)=\mathrm{O}(T^{-2}) \tag{A1}$$

with the initial data (5.17). To solve this problem we use the method which was applied by us earlier in section 3.2.

On the classical trajectory $\boldsymbol{x}(t)=\boldsymbol{x}_0(\tilde{R}(s))+(1/T)\boldsymbol{x}_1(s,\theta)+\mathrm{O}(T^{-2})$ (see formulae (3.12), (3.15) and (3.21)) governing the adiabatic motion of the classical system with the Hamiltonian function $\lambda^{(+)}(\boldsymbol{p},\boldsymbol{x},R(t))$ the polarization vector $\mathscr{B}(t)$ (5.18) admits the following expansion in the parameter T^{-1}:

$$\mathscr{B}(t)=\mathscr{B}(\tilde{R}(s))+\frac{1}{T}\mathscr{B}^1(s,\theta)+\mathrm{O}(T^{-2}) \tag{A2}$$

where

$$\mathscr{B}(\tilde{R}(s))=\frac{e_0}{2mc}\boldsymbol{H}(\tilde{R}(s)) \tag{A3}$$

and

$$\mathscr{B}^1(s,\theta)=\frac{e_0}{2mc}\left[-\frac{1}{c}\frac{\mathrm{d}\boldsymbol{x}_0(\tilde{R}(s))}{\mathrm{d}t}\times\boldsymbol{E}(\tilde{R}(s))+\left[\left\langle\boldsymbol{x}_1(s,\theta),\frac{\partial}{\partial\boldsymbol{x}}\right\rangle\boldsymbol{H}(\boldsymbol{x},\tilde{R}(s))\right)\Big|_{\boldsymbol{x}=\boldsymbol{x}_0(\tilde{R}(s))}\right]. \tag{A4}$$

The solution of equation (A1) is assumed to be an asymptotic series of the form

$$v_\zeta(t)=\overset{0}{v}_\zeta(s,\theta_\zeta)+\frac{1}{T}\overset{1}{v}_\zeta(s,\Xi)+\mathrm{O}(T^{-2}) \tag{A5}$$

where $\Xi=(\theta,\theta_\zeta,\theta_{-\zeta})$ denotes a set of rapid variables, which, in addition to the variables θ, also contains two new variables $\theta_\zeta=T\Phi_\zeta(s)$, $\zeta=\pm1$, corresponding to the spin degrees of freedom of an electron. Substituting equations (A2) and (A5) into equation (A1) and equating the terms with the same order in $1/T$ we obtain the equation defining the functions $\overset{0}{v}_\zeta$ and $\overset{1}{v}_\zeta$:

$$-\mathrm{i}\Phi'_\zeta(s)\frac{\partial\overset{0}{v}_\zeta}{\partial\theta_\zeta}+\langle\sigma, \mathscr{B}(\tilde{R}(s))\rangle\overset{0}{v}_\zeta=0 \tag{A6}$$

$$-\mathrm{i}\Phi'(s)\frac{\partial\overset{1}{v}_\zeta}{\partial\Xi}+\langle\sigma, \mathscr{B}(\tilde{R}(s))\rangle\overset{1}{v}_\zeta=\mathrm{i}\frac{\partial\overset{0}{v}_\zeta}{\partial s}-\langle\sigma, \mathscr{B}(s, \theta)\rangle\overset{0}{v}_\zeta. \tag{A7}$$

After solving the first equation we obtain in the zero approximation (allowing for the condition of 2π periodicity with respect to the variables θ_ζ) that

$$\Phi_\zeta(s)=\int_{s_0}^{s}\mathrm{d}s\,\Omega_\zeta(\tilde{R}(s)) \tag{A8}$$

$$\overset{0}{v}_\zeta(s, \theta_\zeta)=v_\zeta(\tilde{R}(s))\exp[-\mathrm{i}(\theta_\zeta+\mathscr{M}_\zeta(s))] \tag{A9}$$

where the quantities $v_\zeta(\tilde{R}(s))$and $\Omega_\zeta(\tilde{R}(s))$ are given by equation (5.5). The arbitrary real functions $\mathscr{M}_\zeta(s)$ appearing in equation (A9) satisfy the initial condition $\mathscr{M}_\zeta(s_0)=0$ and are to be determined.

We now turn to equation (A7), whose solution will be obtained in the form of an expansion in terms of the eigenvectors $v_\zeta(\tilde{R}(s))$, $\zeta=\pm1$-:

$$\overset{1}{v}_\zeta(s, \Xi)=\sum_\eta b_\zeta^\eta(s, \Xi)v_\eta(\tilde{R}(s))\exp[-\mathrm{i}(\theta_\zeta+\mathscr{M}_\zeta(s))] \tag{A10}$$

Inserting equation (A10) into equation (A11) and multiplying the left-hand side by $\overset{+}{v}_{\zeta'}(\tilde{R}(s))$, we obtain (denoting for convenience $f_\zeta=v_\zeta(\tilde{R}(s))$)

$$-\mathrm{i}\Omega\frac{\partial b_\zeta^{\zeta'}}{\partial\Xi}+(\Omega_{\zeta'}-\Omega_\zeta)b_\zeta^{\zeta'}=\frac{\mathrm{d}\mathscr{M}_\zeta(s)}{\mathrm{d}s}\delta_{\zeta\zeta'}+\mathrm{i}\overset{+}{f}_{\zeta'}\frac{\mathrm{d}f_\zeta}{\mathrm{d}s}-\overset{+}{f}_{\zeta'}\langle\sigma, \mathscr{B}^1(s, \theta)\rangle f_\zeta. \tag{A11}$$

After separating the explicit dependence on the variables θ in the vector $\mathscr{B}^1(s, \theta)$ (equation (A4)), we rewrite it in the form

$$\mathscr{B}^1(s, \theta)=\mathscr{B}^1(\tilde{R}(s))+\sum_{k=1}^{3}[\mathscr{B}_k^1(s)\exp(\mathrm{i}\theta_k)+\overset{*}{\mathscr{B}}{}_k^1(s)\exp(-\mathrm{i}\theta_k)]. \tag{A12}$$

To solve equation (A11) we consider the following two cases:

(i) Let $\zeta=\zeta'$. The requirement for the solution to be bounded with respect to all the variables Ξ leads to a determination of the functions $\mathscr{M}_\zeta(s)$,

$$\mathscr{M}_\zeta(s)=\int_{s_0}^{s}\mathrm{d}s\,\Omega_\zeta^1(\tilde{R}(s)), \tag{A13}$$

where the integrand coincides with equation (5.23). Allowing for equation (A13), the solution of equation (A11) takes the form

$$b_\zeta^\zeta(s, \theta)=b_\zeta(s)-\sum_{k=1}^{3}\frac{1}{\Omega_k}[\overset{+}{f}_\zeta\langle\sigma, \mathscr{B}_k^1(s)\rangle f_\zeta\exp(\mathrm{i}\theta_k)-\overset{+}{f}_\zeta\langle\sigma, \overset{*}{\mathscr{B}}{}_k^1(s)\rangle f_\zeta\exp(-\mathrm{i}\theta_k)]. \tag{A14}$$

The arbitrary functions $b_\zeta(s)$ given here are chosen at the initial moment of time from the condition $b_\zeta^\zeta(s, \theta)|_{s=s_0}=0$.

(ii) Let $\zeta' = -\zeta$. In this case by imposing an additional condition of 2π periodicity of the functions $b_\zeta^{-\zeta}$ with respect to Ξ, we obtain

$$b_\zeta^{-\zeta}(s, \Xi) = \tilde{b}_\zeta(s) \exp(2\mathrm{i}\theta_\zeta)$$

$$- \sum_{k=1}^{3} \left(\frac{\overset{+}{f}_{-\zeta} \langle \sigma, \mathscr{B}_k(s) \rangle f_\zeta}{\Omega_k - 2\Omega_\zeta} \exp(\mathrm{i}\theta_k) - \frac{\overset{+}{f}_{-\zeta} \langle \sigma, \overset{*}{\mathscr{B}}{}_k^1(s) \rangle f_\zeta}{\Omega_k + 2\Omega_\zeta} \exp(-\mathrm{i}\theta_k) \right)$$

$$+ \frac{1}{2\Omega_k} \overset{+}{f}_{-\zeta} \left[-\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}s} + \langle \sigma, \mathscr{B}^1(\tilde{R}(s)) \rangle \right] f_\zeta \qquad \text{(A15)}$$

where the functions $\tilde{b}_\zeta(s)$ are such that the initial condition $b_\zeta^{-\zeta}(s, \Xi)|_{s=s_0} = 0$ is held. Notice that in deriving equation (A15) the assumption that the quasi-classical spectrum (5.9) is non-degeneration was essentially used. So, the validity of formulae (5.22)–(5.24) immediately follows from the results given here.

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